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LINKING PROBLEM SOLVING AND LEARNING CONTENTS: THE CHALLENGE OF SELF-SUSTAINED STUDY AND RESEARCH PROCESSES

Marianna Bosch[*], Carl Winsløw[**]

ABSTRACT

A main difference between the mathematical activity of students and that of researchers is that researchers pursue their mathematical work in a seemingly self-sustaining dynamics of questions and answers, while students rely on teachers to sustain this dynamics. Unlike researchers, students generally do not construct the questions they work on, and do not search, rearrange and question the established contents they need to answer the questions. The basic problem approached in this paper is: could students also engage in a more self-sustaining and complete work with questions and answers? We first present an analysis of four main paradigms of teaching and learning mathematics, based on different approaches to learners' work with questions and answers. We then discuss and exemplify certain principles for self-sustained mathematical activities using Chevallard's Herbartian schema. The access to new external answers and their test against an appropriate experimental milieu is shown to be a crucial bootstrap for the dynamics of research and study processes.

Key-words: problem solving, problem posing, anthropological theory of the didactic, study and research processes, mathematical inquiry.

CONNECTER LA RÉSOLUTION DE PROBLÈMES ET L'APPRENTISSAGE DE SAVOIRS: LE DÉFI DES PROCESSUS D'ÉTUDE ET DE RECHERCHE AUTORÉGULÉS

Résumé – Une différence notable entre l'activité mathématique des élèves et celle des chercheurs est que les chercheurs développent leur activité sur ce qui apparaît comme une dynamique autoalimentée de questions et réponses, alors que chez les élèves c'est normalement le professeur qui alimente cette

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dynamique. À différence des chercheurs, les étudiants ne construisent pas en général les questions qu'ils approchent et ne cherchent pas, ni réarrangent ou questionnent les savoirs ou réponses établies dont ils ont besoin pour répondre aux questions. Cet article aborde le problème suivant: est-ce possible que les étudiants s'engagent aussi dans une dynamique autoalimentée de travail avec des questions et des réponses ? Nous présentons d'abord une analyse de quatre grands paradigmes d'enseignement et d'apprentissage, selon l'approche qu'ils proposent du travail des élèves avec les questions et réponses. Cette analyse s'étend aux paradigmes de recherche en didactique centrées sur le *problem solving* et le *problem posing*. Finalement, à partir de la notion de schéma herbartien, nous discutons et illustrons certains principes et conditions pour le développement d'activités d'étude autorégulées. Nous montrons en particulier comment l'accès à des réponses extérieures à partir de différent media et leur mise à l'épreuve sur un milieu expérimental approprié apparaissent comme un moteur essentiel pour la dynamique des processus d'étude et de recherche

Mots-clés: résolution de problèmes, formulation de problèmes, théorie anthropologique du didactique, parcours d'étude et de recherché, démarche d'investigation mathématique.

CONECTAR LA RESOLUCIÓN DE PROBLEMAS Y EL APRENDIZAJE DE SABERES: EL RETO DE LOS PROCESOS DE ESTUDIO E INVESTIGACIÓN AUTOREGULADOS

Resumen – Una diferencia notable entre la actividad matemática de los alumnos y la de los investigadores es que los investigadores llevan a cabo su actividad en lo que aparece como una dinámica autoalimentada de preguntas y respuestas, mientras que en el caso de los alumnos, es generalmente el profesor quien alimenta esta dinámica. A diferencia de los investigadores, los alumnos no suelen construir las cuestiones que abordan y no buscan, reorganizan ni cuestionan los saberes o respuestas establecidas que necesitan para responder a las cuestiones. Este artículo aborda el problema siguiente: ¿es posible conseguir que los estudiantes se involucren en una dinámica autoalimentada de trabajo con cuestiones y respuestas ? En primer lugar presentamos un análisis de cuatro grandes paradigmas de enseñanza y aprendizaje basados en distintas maneras de abordar el trabajo de los alumnos con preguntas y respuestas. Se discuten e ilustran entonces algunos principios para el desarrollo de actividades matemáticas autorreguladas utilizando la noción de esquema herbartiano. Mostramos en particular cómo el acceso a nuevas respuestas externas y su contraste con un medio experimental apropiado aparece como un motor crucial para la dinámica de los procesos de estudio e investigación.

Palabras-claves: resolución de problemas, formulación de problemas, teoría antropológica de lo didáctico, recorridos de estudio e investigación, indagación en matemáticas

1. DEALING WITH PROBLEMS AND CONTENTS IN A DYNAMIC PROCESS

Raising questions, formulating problems and solving them constitute a crucial driving force in the development of mathematics. However, mathematical activity cannot be reduced to problem solving. Another important dimension of this activity is the organisation and study of the results obtained by the resolution of problems.

The French mathematician and epistemologist Georges Bouligand (1889-1979) proposed to describe the development of mathematical activity as a dialectic between solving problems and the elaboration of what he called the *synthesis*:

> To get a general idea of mathematical thought according to its works, one must first examine mathematical activity: on the one hand, the problems, which allow to determine an unknown element under precise conditions; on the other hand, the synthesis which keeps tracks of new problems and assembles results known to coordinate an inventory of methods, operations. From this comes a specific schema for the historic development of the deductive sciences, which better underlines its dialectic character, involving both logics and the process of mathematizing large domains. (Bouligand, 1957, p. 139; our translation)

It is thus in the interaction between problems and syntheses that the dialectic takes place. The synthesis serves to select, organise and connect the results obtained during the complex process of solving a problem. This work usually generates new questions to be addressed, and new ways of addressing old open questions:

> Paying alternately attention to problems and the global synthesis, the diligent researcher will use his available freedom to enable one of these tasks support the other (ibid., p. 124). His attention can hardly be confined in an exclusive manner to the side of problems or of synthesis; one has seen, in fact, numerous occasions to pass from one to the other. (…) In most cases, it is under the influence of an adequate group of problems that one proceeds towards a satisfactory synthesis, as grouping is the process through which generality is found. In view of the fact that a synthesis under accomplishment, or being accomplished, often gives rise to new problems, the aforementioned cycle can be repeated several times (ibid., pp. 134f).

The traditional way to disseminate mathematical knowledge is based on the transmission of *syntheses*. Mathematical knowledge, as it has been selected and organised for schools, is structured as an already finished product, using a rich terminology to describe the main notions, results and techniques, and leaving little room for connecting the questions that did or could motivate their construction.

A lot of innovative research has grown out of locally successful attempts to change this situation, often based on pedagogies that emphasise the art of *problem solving* more or less exclusively (e.g. Pólya, 1945; Schoenfeld, 1985). Moreover, research on problem solving has also addressed the difficulties in combining, in a coherent way, the approach of open questions with more traditional school objectives (e.g. Schoenfeld and Kilpatrick, 2013). In fact, the conditions needed to manage the transmission of specific knowledge organisations (the *synthesis* in Bouligand's terms) are not the same as those required to teach and learn how to solve open-ended problems.

The dualism between problems and synthesis corresponds to the global challenge in mathematics education of how to articulate content-defined with problem-based teaching and learning processes, and, indeed, the potential *dialectics* between the two. What role could this dialectic play in the school context, and what means are needed to realise it? Can it take on a similarly dynamical form for students as for mathematicians, with a self-sustained interest and development of problems and syntheses? These are the broad questions that we wish to address in this paper. They include what we can call *problem management* in the classroom, that is, the capacity for a group of students led by a teacher to raise problematic issues, select some of them while rejecting others, reformulate the chosen ones to make them more easily approached, derive other related issues, apply new transformations, connect some to others so as to make the questioning evolve, etc. - with the intermediate aim of obtaining an organised body of knowledge ready to be used for further exploration, including to address new issues and propose new problems to be solved.

In particular, we will develop a proper didactic formulation of the dialectic between problems and synthesis proposed by Bouligand, while taking into account the results and limitations of previous research on different aspects of this dialectic, such as the autonomous solving or formulation of problems by students (Sec. 2.3 and 3) or the engagement of students in genuine "study and research processes" (Sec. 4.3). In particular, our own research in the last-mentioned context (e.g. Barquero, Bosch & Gascón, 2013; Barquero & Bosch, 2015; Winsløw, Matheron & Mercier, 2013) has led us to consider the following questions and challenges:

- Considering that traditional teaching is more based on the transmission of "syntheses" than on the implementation of problem development activities, what *didactic strategies* have been or could be proposed, theoretically and empirically, for engaging students in a *sustained, long-lasting* development of interacting problems and syntheses? What are the essential conditions – perhaps neglected in past efforts and research – to ensure that students may pursue such an activity based on their own knowledge and the results of their work, without the necessity of the teacher's constant interventions?

- What *institutional constraints* related to the realisation of such a self-sustaining development have been observed, for instance, in the contexts of secondary schools and tertiary education frameworks? What parts of these constraints are due to the didactic transposition process and the type of mathematical resources made available to school agents (teachers and students)? What strengths and shortcomings, related to these constraints, can be observed in various approaches to the didactic strategies engaging students in developing mathematical problems and syntheses in a coherent way?

We will primarily treat the first group of questions, while some further perspectives on the second group are given in the final section.

2. TWO DIMENSIONS IN PEDAGOGICAL PARADIGMS

We use the Anthropological Theory of the Didactic (abbreviated ATD; see e.g. Chevallard, 1999; Bosch et al., 2011) to approach the aforementioned questions. In this framework, the duality between problems and syntheses described by Bouligand is conceived as a *dialectic between questions and answers*. It is not restricted to mathematical activity but is considered the core of the development of all kinds of human knowledge and practices.

The general schema is the following. Human knowledge arises from the study of important and problematic *questions*. In order to provide answers to a given question, we make use of a variety of materials, some of which are in turn answers previously elaborated by others to the same or to related questions. These answers are the synthesis of previous research processes and the outcome of a specific organization of the results obtained in the process. As these syntheses become shared knowledge, they can function as resources for further study, giving rise to new questions. In particular, an *answer* is understood as established knowledge, which becomes public through a variety of *media* (books, journal articles, conference talks, teachers, web tutorials and so on). While answers are sometimes considered final - at least for a while - they can also be important resources for posing or treating new questions. Notice that the usage of the terms *questions* and *answer* can be quite restrictive with respect to general usage (certainly, phrases such as “How do you do?” are grammatical questions but are not problems in any common sense); we shall maintain this restricted usage of the terms throughout the paper.

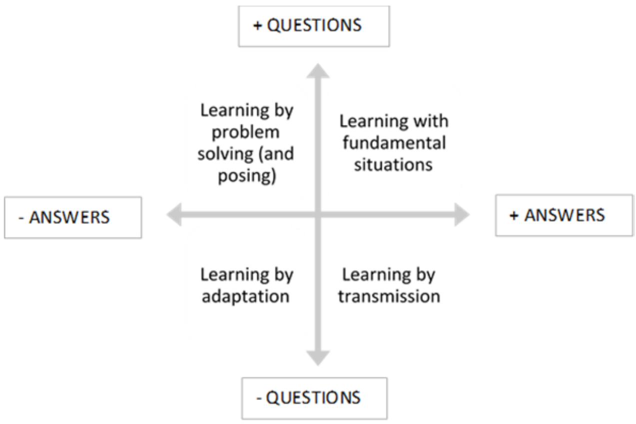


Figure 1. Four pedagogical paradigms defined by the role given to questions and answers

The balance and connections between the pursuit of questions and the study of available answers leads to a way to analyse and characterize major pedagogical approaches. We illustrate these two dimensions in Figure 1. The vertical axis indicates the degree to which the teaching and learning process involves explicitly formulated *questions*. The horizontal axis indicates the degree to which it is based on previously established answers accessed during the study process as contents to be learned.

Each pedagogical approach can be located in a quadrant of figure 1 and appears in historical or more recent paradigms in the teaching and learning of mathematics. We will use them to specify what is needed and what is currently missing in the implementation of school mathematical processes sustained by a dynamics based on the dialectic of questions and answers.

2.1. Learning by transmission

The study of “answers without questions” may appear at first sight bizarre, but as observed in the introduction, it is the most classical, and still most prevailing, organisation of mathematical education. It consists in the direct transmission of syntheses - in school settings, presented in heavily transposed forms through textbooks and other media. We are all very familiar with this pedagogical schema, for instance through the format of lectures (in which lecturers act as media). "Teaching by telling dominated the Antique world, at a time when manuscripts were scarce and most teachers and learners had to rely on memory:

> [In Ancient Greece, as] in the Ancient East, teaching remained rudimentary. (…) Tradition having decided what to be taught (…) the teacher's job was simply to go on repeating the same thing over and over again until the child saw the light. (Marrou, 1956, p. 158).
>
> Roman teaching methods were as Greek as the Roman Syllabus. They were entirely passive. The most highly prized qualities were a good memory and powers of imitation. (ibid., p. 272)

It remains a very common form of teaching and learning mathematics to simply study given answers through oral and written media (both increasingly represented and transmitted digitally), with little concern for the questions that gave rise to the specific synthesis

2.2. Learning by adaptation

Teaching and learning mathematics with *no* explicit questions and answers being raised may sound even odder than the paradigm of studying answers without questions. However, in some sense it is even older than the antique practice of learning by transmission: the rudimentary forms of mathematical practices like counting and ordering – as well as innumerable other human capabilities – are certainly acquired in material and practical contexts, with observation and imitation as the main sources of learning. So the pedagogy founded on transmission can be seen as a departure from a more tacit pedagogy of adaptation to the environment, where the main difference is the presence in the former of a discourse to support, explain and justify the practice imitated.

An early and famous expression of this general principle is found in Rousseau's *Émile*, and continues in the broad lineage of thought represented by Montessori, Brunner and Piaget. A fundamental argument for privileging autonomous learning by adaptation to the environment over learning based on direct instruction is that the former leads to knowledge of a somehow higher quality:

> Let us transform our sensations into ideas, but do not let us jump all at once from the objects of sense to objects of thought. The latter are attained by means of the former. Let the senses be the only guide for the first workings of reason. No book but the world, no teaching but that of fact. The child who reads ceases to think, he only reads. He is acquiring words not knowledge. (Rousseau, 1762, pp. 203-204).

In the context of mathematics teaching, the ideal of learning that takes place spontaneously from direct confrontation with phenomena continues to thrive. The classical idea that concrete materials can support primary schools students to "develop new mathematical concepts at first hand, i.e. directly from experience" (Dienes, 1963, p. 124) can be recognised in many recent uses of mathematical software as a means to create a kind of mathematical nature which, at least apparently, let the students experience and interact with mathematical objects much as they experience plants in nature.

In summary, this paradigm recommends that the teacher refrains from proposing answers and even questions, and instead that the teacher exposes the learners to a rich environment and maximize their own initiative in this regard, with the presumed result that the questions and answers found will be based in personal reflection rather than repetition.

2.3. Learning by problem solving

The fundamental concern that the learner takes an active role in the construction of his knowledge can of course be pursued in less radical ways than refraining from imposing any questions or answers on the learner. Since the early 20th century, beginning with figures like Dewey and Pólya, a widespread strategy - with many variants - has been to expose students to *problems,* while leaving learners with the responsibility to produce and usually also to validate their *answers.*

There is no shortage of statements by mathematicians and educators that solving problems is an important - if not crucial - part of doing and learning mathematics. Even if there is no common definition of what exactly defines this activity, a number of scholars such as Pólya (1945) and Schoenfeld (1985) have provided widely accepted characterizations of it, including the non-triviality of the question worked on, and the novelty of results or methods. For the mathematics researcher, what is known to the mathematical community (more or less clearly delimited by accessible publications or communications) is the baseline for novelty and challenge. In the educational context, these criteria are normally understood relatively to the person or group engaging in a search for answers to the question mathematics education (see e.g. Schoenfeld, 1985).

Indeed, *problem solving* as a model and goal of students' mathematical learning has been intensively developed, theorised and explored over the past century. While the definition of what constitutes a "problem" remains somewhat blurred, activities called "problem solving" usually involve presenting students with one or more questions – considered by the poser as problems for the students

– and asking students to elaborate answers to the question. Both exercises and challenging problems occur in specific contexts which, perhaps, are not very important in themselves; it is a strong underlying assumption of learning by problem solving that one gains *more* from solving a problem than getting to know the answers obtained.

2.4. Learning with fundamental situations

The Theory of didactic situations (TDS, Brousseau, 1997) provides a framework where the dialectic between questions and answers appears in an explicit way. Students are faced with a set of adidactic situations consisting of an experimental environment (a *milieu*, i.e. a set of objects and relationships which are familiar to the students and which do not *per se* instruct them) and one or more questions posed by the teacher. The students learn by jointly elaborating a solution to the problem through their interaction with the milieu. The teacher helps them in the process by making the milieu evolve by appropriate adjustments of the *didactical variables*, but without providing any element of answer. This, *per se*, is similar to problem solving. However, the sequence of situations is supposed to be designed so as to enable students to carry out the construction of specific new mathematical knowledge associated with the questions raised within the situations. At the end, when the solutions have been carried out, formulated and validated by the students, it is the responsibility of the teacher to relate them with existing mathematical knowledge (*institutionalisation*: establishment of answers which the institution aims to disseminate). In a so-called fundamental situation (ibid., p. 47), this mathematical knowledge at stake is the *only* possible answer, given the milieu devolved by the teacher.

While the questioning dimension of adidactic situations seems clear, what can we say about the study of previously elaborated answers, syntheses or what Brousseau calls the "established knowledge"? This is where the teacher (and also the didactician) intervenes. TDS postulates that any piece of mathematical knowledge can be modelled by means of one or more fundamental situations. In the example, the situation of the puzzle is fundamental for a piece of knowledge which is usually indicated by terms such as similarity and linear mappings, and which is also associated to "the difference between an arithmetical and a geometrical increase", etc. Each sequence of adidactic situations takes its place in a larger set of sequences, constructed to organise a whole domain of mathematics.

Brousseau (1997, p. 22-23) presents what we call the dialectic between questions and answers, or problems and syntheses, through a comparison of “the work of the mathematician” and “the work of the teacher”. The mathematician, according to Brousseau, starts from questions and, through painstaking research as well as study of earlier works, may succeed to produce a set of answers. She then “depersonalizes, decontextualizes and detemporalizes her results as much as possible” (ibid., p. 22), in order to detach the results from the fortuities of the discovery process, to achieve maximal generality, and connect them to pre-existing, warranted answers. On the other hand,

> The teacher’s work is to some extent the opposite of the researcher’s; she must produce a recontextualization and a repersonalization of the knowledge. It must become the student’s knowledge, that is to say, a fairly natural response to relatively particular conditions, conditions that are essential if she is to make sense of this knowledge. Each item of knowledge must originate from adaptation to a specific situation… (p. 23)

In short: the teacher *begins* with answers and constructs questions, along with appropriate milieus, which are adapted to the specific conditions of the school, the students’ capacities and so on. We notice here that while TDS emphasizes the construction of questions, the culturally given answers do not remain in the shadow but are present both in the construction of situations and in the final phase of institutionalisation. It is in this sense that the model of “learning in fundamental situations” includes, at the same time, genuine problems and explicit syntheses, in a coherent way.

What is also clear, though, is that in the TDS model of learning, just as in the model of learning through problem solving, the initiative regarding the *posing of questions* remains, largely, with the teacher. This, in fact, makes it difficult to sustain longer processes of student inquiry, in which both answers and questions and elaborated in a coherent logic. While students are given some initiative in the production of answers, teachers are needed to make the process go on from the production of personal answers to a synthesis into institutionally recognized answers and in proceeding from there to new, meaningful and challenging questions to study. Thus, the dialectic of questions and answers does not lead to a self-sustained process of student work because teachers must choose and formulate the main questions for students to work on, and to link the students’ productions to pre-established answers (a step often skipped in problem solving). This also means that students’ work in fundamental situations misses an important element of the mathematicians’ work:

> We do mathematics only when we are dealing with problems – but we forget at times that solving a problem is only a part of the work; finding good questions is just as important as finding their solutions. A faithful reproduction of a scientific activity by the student would

> require that she produce, formulate, prove, and construct models, languages, concepts and theories; that she exchange them with other people; that she recognize those which conform to the culture; that she borrow those which are useful to her... (Brousseau, 1997, p. 22)

The four pedagogical approaches described in a necessarily simplified way lead us to formulate the following postulate. To enable a more self-sustained dynamics of the kind foreseen by Bouligand, mathematical activities currently implemented at school are still missing three main elements: (1) students engaging in "finding good questions" and considering them as the starting point of actual or potential inquiry processes; (2) students engaging autonomously in "finding potential answers", which includes the study of culturally institutionalised pieces of knowledge of presumed pertinence to the questions studied; and (3) students engaging in "finding rich milieus" to test the validity of the potential answers for the questions approached and to nourish the elaboration of final responses.

In the next section, we shall examine the results of long-standing research programmes specifically focused on the "finding good questions" part of (1). Then, in section 4, we consider a proposal based on our recent research in the ATD which aims at bringing the three elements into play through a dynamics of study and research.

3. RESEARCH ON STUDENTS' PROBLEM POSING

The main difference between the paradigm of "problem solving" (Section 2.3) and the paradigm of fundamental situations (Section 2.4) is the epistemic dimension of the latter, implying that a set of problems are constructed together with appropriate milieus which, properly sequenced, will at least theoretically allow the students to reconstruct *a well-defined set of answers* which the teaching institution intends the students to acquire. But while the aims of problem solving are usually more modest and implicit in terms of ensuring the specific mathematical answers elaborated, they are more ambitious in terms of the autonomy in which students get to elaborate and validate their answers.

Both approaches are affected by the observation of Kilpatrick (1987) that teachers and students alike take for given that the problems to be worked on by students have to be brought about, at least in an initial form, by teachers or more generally by the institution which engage students in problem solving:

> …almost all of the problems that a student encounters have been proposed, and formulated, by another person - the teacher or the

> textbook author. In real life outside school, however, many problems, if not most, must be created or discovered by the solver, who gives the problem an initial formulation. (Kilpatrick, 1987, p. 124)

Naturally, "one can think of problem solving as consisting of successive reformulations of the initial problem" (ibid., p. 125). But it is well known that the school practice outlined in the previous quote tends to generate a reduced, deformed image of mathematical work: there is always just one good answer, and the criterion for goodness is that it is the teacher's one. By contrast, as observed by Brown and Walter (1983, p. 5) in a classical treatise on teaching problem posing:

> Given a situation in which one is asked to generate problems or ask questions – in which it is even permissible to modify the original thing – there is no right question to ask at all.

Still, the challenge remains: is it feasible and desirable to have students take a more active role in identifying or formulating the questions they work on, and thus to make their activity more akin to what Kilpatrick considers the situation "in real life outside school"?

3.1. Where do mathematical problems come from?

The activity of mathematician researchers has often been taken as a model – or at least a metaphor – of ambitious programs for students' work with questions and answers. So, what is the source of the questions they work on? Do they formulate the questions by themselves?

Certainly, famous problems which have been around for some time can present themselves as ready made to the professional mathematician, much as an exercise appears the student (except for the institutional warrant that exercises do have known answers). Such famous problems can be seen as the product of collective work.

Halmos, after a review of more or less famous collections of problems published by mathematicians like Hilbert and Pólya, states his belief that

> problems are the heart of mathematics, and I hope that as teachers, in the classroom, in seminars, and in the books and articles we write, we will emphasize them more and more, and that we will train our students to be better problem-posers and problem-solvers than we are (Halmos, 1980, p. 524).

On the other hand, as was pointed out by Bouligand (second quote given in the introduction), new questions may arise naturally, for the mathematician, from acquired and appropriately synthesized answers - such as whether some condition can be dispensed with, if the converse of a proven implication holds true, etc.

Thus, identifying, sharpening and developing mathematical questions is both an integral and important part of mathematical activity. The idea that this should also be done in ordinary school settings has motivated several scholars to examine the possibilities and benefits of "problem posing" as part of students' mathematical activity. In the next sections, we shall examine one major direction of research into this matter.

3.2. From teachers' to students' problem posing

A considerable research effort has been devoted to implement and investigate students' *problem posing* as a distinct mathematical activity, with close links to the paradigm of *problem solving* (as described in Section 2.3). The approach is closely related to recent American "standards" for mathematics education. For instance, the NCTM standards from 2000 state, in the context of problem solving:

> A major goal of high school mathematics is to equip students with knowledge and tools that enable them to formulate, approach, and solve problems beyond those that they have studied. […] They should have opportunities to formulate and refine problems because problems that occur in real settings do not often arrive neatly packaged. Students need experience in identifying problems and articulating them clearly enough to determine when they have arrived at solutions. (NCTM, 2000, p. 335).

One can get a reasonable overview of recent "problem posing" research through the special issue of *Educational Studies in Mathematics* (volume 83, issue 1) published in May 2013. The editors emphasize, as do several of the papers, that this research topic is relatively recent and largely "outside the vision and interest of the mathematics education community", so that at present "the field of problem posing is still very diverse and lacks definition and structure" (Singer, Ellerton & Cai, 2013).

In fact, this lineage of research has at least 30 years of history, with the works of Kilpatrick (1987) and Brown and Walter (1983) as some of the early impulses.

In outline, the following main types of problem posing tasks are experimented in this research tradition:

(1) a standard exercise (or problem), modified by simply omitting the question and replacing it with a request for the subject to formulate one or more question, for example:

> Ann has 34 marbles, Billy has 27 marbles, and Chris has 23 marbles. Write and solve as many problems as you can that uses [sic] this information. (Singer, Ellerton, Cai & Leung, 2011, p. 150);

(2) a standard exercise (or problem) followed by a request for subjects to formulate a similar or related question, for example:

> *Solve the following system of equation:*

$$\begin{cases} x + y = 8 \\ 5x + 7y = 50 \end{cases}$$

Answer: x = y = Show your work.
Write a real life problem that could be solved using the above system of equations. Be specific.
(Cai, Moyer, Wang, Hwang, Nie & Garber, 2013, p. 63)

(3) more open situations, with some elements of situation or phenomenon described, based on which problems are to be formulated (an example is shown in Figure 2; this example is used in several papers and in fact this kind of situation is much rarer in the literature).

Imagine Billiard tables like the ones shown below. Suppose that a ball is shot off at a 45^0 angle. In Table 1, the ball travels on a 4×6 table and ends up in a pocket B, after 3 hits on the sides. In Table 2, the ball travels on a 2×4 table and ends up in pocket B, after 1 hit on the side. In each of the figures shown below, the ball hits the sides several times and then eventually lands in the corner pocket.

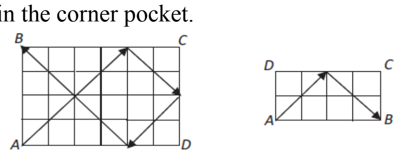


Table 1 Table 2

Based on this situation, pose and write down as many interesting mathematical problems as you can. You may drop, change or add the contitions. The other students will be asked to solve the problems you

Figure 2. The "Billiards Table Task" from Kontorovich et al., 2012, p. 154.

Researchers generally confront students or teachers individually with one or more tasks of the above types, and interpret the outcomes in view of different aims, for instance to investigate connections between problem posing and problem solving performance of individual students (Cai et al., 2013), to compare students' and teachers' problems arising from the same situation (Singer et al., 2011, p. 151), or to investigate the processes through which small groups of students elaborate problem formulations (Kontorovich et al., 2012).

With few exceptions, it is clear, albeit implicit, that "problems" are to be understood in the sense of "school problems" which are texts formed by a *descriptive part* which furnish a minimal set of conditions and information that are sufficient to answer *one or more questions* (appearing as the second part of the text), formulated with reference to the descriptive part. For instance, saying that problem solving capacity in the ordinary school setting (cf. Section 2.3) can be diagnosed or furthered through work with problem posing (Cai et al., 2013) is clearly associated with the idea that respondents are meant to formulate word problems of the kind that appear in the school setting. Frequently this is even explicit in the wording of the request of the problem posing tasks, e.g. “For his students’ homework, Mr. Miller wanted to make up three problems (…) Help Mr. Miller…” (Singer et al., 2011, p. 157). The frequent request to provide solutions together with the problem may encourages students to produce easy school exercises rather than questions which are really problematic to them. But students seem to interpret “good tasks” as “school like tasks” even when they are not asked to provide solutions.

Much problem posing research thus operates with a relatively limited notion of problem and, consequently, of problem posing, reduced to a text with rather specific features: it must be *self-contained* in the sense that it can be understood based on expected routine knowledge of the respondents (without further study); also, it does not refer to a wider set of problems, an ongoing investigation or the like. This fits with the prevailing timescale of the research experiments, which usually involves punctual exposition to problem posing tasks, with little account of respondents’ previous or normal mathematical activity. Problem posing, thus, tends to become a somewhat isolated experience, and it is evaluated based on more or less implicit criteria of quality (e.g. solvability for students, sufficient information given, etc.) that may be applied by a teacher who examines problems in a textbook.

Even if it appears to be a widespread ambition that posing problems becomes a more integrated part of students mathematical work in school (cf. the NCTM standard quoted above), the investigation of actual “classroom instruction where students are engaged in problem-posing activities” appears to be more or less virgin territory for the research programme of problem posing (Singer et al., 2011, pp. 157-158; for an exception, see Kontorovich et al., 2011). This programme has thus, so far, mainly been engaged with fundamental research in a framework mainly oriented by cognitive theory, where, for instance, students are exposed individually to problem posing tasks, their productions are analysed independently from institutional conditions and without any epistemological analysis of the specific mathematical context of the problems.

3.3. Where can problem situations come from?

Before going further into didactical designs which allow or encourage students to pose questions as part of a learning process, let us return to our initial question in this section: “Where do problems come from?” The idea of *problem posing situation* is, potentially, much richer than the rigid genre of *problem posing tasks* which were considered above, and in which the drive to formulate questions comes from an explicit request to *make up* problems of a school-like nature.

As Fabre (1997, p. 49-50) observes, the term *problem* (πρόβλημα, in ancient Greek) has three etymological roots: obstacle (*problema*: something placed in front of you); something that stands out (*problema*: promontory); initiative, project (*proballein*: to cast forward). The first two are semantically close except for the interpretation: something that stands out may be simply noticeable or significant, or it may be considered an obstacle to be overcome. The last sense is more active: formulating, planning and acting, rather than just noticing. At any rate, we are reminded by these roots of the word that a problem does, indeed, not arise out of the blue, and that *posing a problem* presupposes the noticing of something significant in a situation, and the identification of something that calls for action in the future. In the language of Bouligand (cf. Introduction), identifying and proposing problems must be based on some synthesis or overview of the situation, without which one cannot notice significant elements or obstacles. A problem posing situation must engage participants in identifying one or more “problems” (what they don’t know) on the background on what they know (established knowledge, synthesis). It must thus contain several elements known to the participants but also some salient features that they could notice on this background, but not immediately account for: “To ask a question, one must know enough to know what is not known”, as observed by Miyake & Norman (1979) in an early empirical study of problem posing. One may add that a certain experience with questioning - including some techniques and notions related to formulating a problem - would also be of value; we return to this and other points on “teaching problem posing” in the next subsection.

In summary, the analysis and design of problem posing situations cannot fruitfully ignore the students’ history with the subject at hand, or expect that results from punctual expositions of informants to more or less arbitrary and trivial situations will lead to quality problem posing.

3.4. Teaching to pose problems

So far, we have mainly considered the study of problem posing situations and their outcomes, rather than the didactic knowledge (practical as well as theoretical) needed to design and implement them, possibly as an element of a wider didactic design.

In the literature on mathematical problem posing, we find mainly two types of studies: on the one hand, teachers' problem posing performance is investigated with much the same methodologies as in experiments with students, and sometimes the two groups are compared (e.g. Singer & Voica, 2013); on the other hand, equipping teachers' with ready-made problem posing tasks and tools for coding students' problems, researchers investigate their use of these tools in teaching (e.g. Leung, 2013).

The literature on problem posing in the context of science proposes more practical guidelines for teachers who wish to cultivate problem posing among students. Chin (2002, pp. 157-164) suggests five major tasks for teachers in this regard, along with a number of research based techniques. Firstly, it is important that students learn to distinguish and articulate certain *types* of question – to begin with, questions are "investigable" and "non-investigable". Among the first kind, she lists 10 categories, many of which are specifically relevant to science teaching (e.g. cause-and-effect, design-and-make). Secondly, Chin insists that students need to be taught a *language* to describe the techniques of problem posing and solving, which includes terms such as "variable", "hypothesis", "wording" and so on, in order to control more explicitly the type of the question developed and in particular to ensure its "investigability". The third set of advices concerns the role of teachers as guides to problem posing, for instance by using "productive questioning and variables can to help students transform a non-investigable question to an investigable one". As a fourth proposal, teachers can deliberately construct situations that are familiar to the students but in which they can also identify questions that are of interest to them (cf. Section 3.3). Finally, teachers must create an environment that "fosters question asking", such as "anomalous happenings and materials that do unexpected things". We notice here, once again, the absence of using external media as an explicit and developed resource in the inquire process.

Similar guidelines are found throughout the literature of "inquiry based science education" which, on this account, seems richer than the "problem posing" literature related to school mathematics. Of particular interest to us is the idea of *driving questions* for project based work in science, as developed by Krajcik and Czerniak (1999, Chap. 3). A driving question should be able to trigger students investigations “over time” and “in great detail”, as they formulate sub-questions, conduct experiments, formulate and solve sub-questions, produce further questions etc.; in line with Chin’s suggestions, the interest of students is mainly sought through the perceived relevance of the question to “students’ lives”, while the question must also be “rich in science content/concepts” and “help students link science concepts”; both of the last requirements would, in our reading, call for deeper epistemic analysis than we actually find in the book.

In the next section, we shall introduce and exemplify a set of theoretical tools aimed to recover, in the context of mathematics education, a sustained dialectic of questions and answers in students’ activity, while maintaining and strengthening the importance of epistemological analysis of both questions and answers, as they appear in media which could serve as resources for students’ work.

4. STUDY AND RESEARCH PATHS AND THE DIALECTIC OF QUESTIONS AND ANSWERS

At the end of section 2, we pointed out three strongly related elements of mathematical activity which seem to be insufficiently engaged in by students in our schools. The first of these elements is *students engaging in the coherent, nontrivial work of finding questions and trying to solve them*. The literature on problem posing (section 3) brings about only a partial approach since it restricts the finding and solving of “good questions” to the case of relatively simple, school-type problems. A further step is done by the contributions of research on inquiry-based teaching, both in mathematics and science education; they confirm and enlarge the results on problem posing considering activities driven by open initial questions leading to a richer and more well-articulated notion of how students could engage in exploring questions as well as existing answers, hypothesize and experiment, and elaborate new pieces of answers. However the proposals take the form of exceptional additions to the standard curriculum, for instance in the context of European projects (e.g. Artigue & Blomhøj, 2013).

The second and third elements identified at the end of Sec. 2 concern the missing link between, on the one hand, learning how to pose and solve problems and, on the other, being able to profit from the huge amount of knowledge elaborated by our predecessors to address the same or similar problems. In other words, the process of finding “already elaborated answers” and “appropriate milieus” to nourish the study of problematic questions remains in standard school practice, the teachers' responsibility. For students, it therefore does not appear explicitly related to problem solving (or posing) processes.

We now present our own proposal to deal with these substantial shortcomings in common students activities within school mathematics. It relies on the latest developments of the Anthropological Theory of the Didactic (Chevallard, 2015) and the results of our own investigations in the implementation and analysis of school teaching processes based on *study and research paths (or courses)* (Barquero et al., 2013; Barquero & Bosch, 2015; Jessen, 2014; Winsløw et al., 2013). We first introduce the notion of *Herbartian schema* that will help describe, from a broad point of view, the linkage between solving questions and studying answers in inquiry processes - including, as a particular case, school teaching and learning activities. We illustrate this use of the schema by a relatively detailed analysis of a question related to the "Billiards Table Task" (Figure 2). We argue that the consideration of previously established knowledge appears to be a necessary condition for the sustained realisation of problem posing, as a kind of bootstrap of the process. The conjectures advanced in section 3.3 lead to some design principles that have been experimentally used in different school settings at secondary and university level, while many complex issues remain to be studied (Section 5).

4.1. A schema to describe the process of inquiry

Referring to the German pedagogue J. F. Herbart (1776-1841), Chevallard (2012) introduced the notion of *Herbartian attitude* to indicate a "receptive attitude towards yet unanswered questions and unsolved problems, which is normally the scientist's attitude in his field of research and should become the citizen's in every domain of activity" (p. 7). He also introduces a symbolic representation of the inquiry processes which follow from this attitude, and we now outline a version of this, adapted (and reduced) for our purposes.

When a person or group of people take interest in a question Q and decides to pursue it, a dialectic of questions and answers emerges. Two different kinds of answers can be distinguished. On the one hand, there might exist established pieces of knowledge and know-how which have been produced by other people in the past to give answer to similar questions and that seem to be relevant to the question under study. These answers are products (or, in the terms of Bouligand, syntheses) of human work and culture, more or less organised and structured, named and labelled to be identified and referenced. They are marked by a "stamp" symbol ($A^{\Diamond}$) to indicate that they have a label and some kind of warrant in the cultural organisation of knowledge, enjoying at least some level of recognition.

However, not all questions can be solved directly by using a set of established answers $A_k^{\lozenge}$. Frequently, some work of combination, adaptation, deconstruction and reconstruction of established answers is necessary to obtain a satisfactory response or solution to Q. Even so, one may not be able to answer the question fully, so that Q is split into some derived questions Q_1, Q_2, Q_3, etc. which become the starting point of new paths to follow. The study of Q thus consists in a sinuous course paced by an arborescence of derived questions and partial answers that ends up in the building of a provisional final answer to Q – including here, in the worst case, the admission of an impossibility to solve Q under the available conditions.

This provisionally final production is indicated as $A^{\heartsuit}$ to indicate its place at the heart of the inquiry process: we study a question to provide an answer, as the target of the process. Moreover, $A^{\heartsuit}$ is at least initially the inquirers' personal answer to Q.

A clear difference between the school and the research dialectic between questions and answers is, indeed, the posterity of answers $A^{\heartsuit}$, that is, the work done with the problem solutions after they are provided. At school, solutions produced by students are rarely considered as productions deserving special attention for being used in the future, to solve other problems or (even less!) to help raise new questions. The pieces of answers $A^{\heartsuit}$ provided are rarely integrated into syntheses $A^{\lozenge}$ and while elaborate procedures of validation do exist in the school context, the main concern is usually to assess the extent to which students' productions correspond to the official labelled answers that are supposed to be learned. As a contrast, in research work, informal solutions $A^{\heartsuit}$ to specific problems Q are usually the raw material of the elaboration of new syntheses $A^{\lozenge}$: they are submitted to a complex process of validation before being summarized or subsumed into a broader set of results, properties and notions that relate to the new established and warranted answer $A^{\lozenge}$. The dialectic between questions and answer thus also includes the process of transforming the personal (and often collective) answer $A^{\heartsuit}$ obtained into public and identifiable bodies of knowledge $A^{\lozenge}$ that can in turn feed into further study and research processes (which is crucial to the idea of sustained or boot-strapped inquiry).

The process starting with the consideration of a question Q and the production of a (usually) provisional answer $A^{\heartsuit}$ is named, in ATD, the *Herbartian schema* (Chevallard, 2008, 2012). It is aimed at providing a broad perspective on school inquiry processes, and to locate them within the extensive class of research and study processes that exist in our societies: research investigations, police or journalistic inquiries, medical diagnostic, company consultancy, and also the minor activities of searching for quick answers by asking experts, professionals or performing a web search.

Besides the initial *generating* question Q and the inquirers' proper answer $A^{\heartsuit}$, the Herbartian schema includes two main elements linked through a dialectic. The first element are the hallmarked answers $A^{\Diamond}$ that the inquirers should search and access through different *media*, that is, through any means with a clear didactic intention, aiming at disseminating knowledge and information to a given target: a textbook, a treatise, an encyclopaedia, a journal, a lecture, a film, an on-line video, a webpage, etc. are examples of *media* In the Herbartian schema, the reception of answers is not necessarily a passive activity. Media need to be constantly questioned and, once a piece of work is accessed, it is necessary to *study* it and check it against the second element of the dialectic: that of *milieu,* understood much as in the Theory of Didactic Situations (Brousseau, 1997). It corresponds to a set of known and established cognitive and material tools which are available for the inquiry process and which are stable enough to act as "a piece of nature", without any didactic intention towards to piece of knowledge considered. We notice that due to the relative absence of explicit questions in most synthetic presentations of $A^{\Diamond}$, it normally takes considerable work to turn answers found in media into elements than can be tested against the milieu. Once accepted, the answers may then be integrated into the milieu.

There are always several possible ways to approach a question Q, depending on available media and milieus, the established answers accessed and, especially, the *derived questions* considered. Many different *research and study paths* (or *courses*) can be followed, during which new derived questions are constructed. With each new question comes the need for media with new established answers, which in turn calls for further elaboration to integrate the answers into milieus for the questions worked on. We notice here that a central motor is the generation of new questions (by derivation from earlier questions, and with the support of established answers found in media). A summary of the process can therefore be given by a tree diagram of research and study paths with questions as vertices and edges representing answers or derivations leading from one question to another (see Figure 4 for an actual example of such a structure, which we explain further in the next section). A symbolic representation of the Herbartian schema can also be given as follows:

$$[S(X; Y; Q) \text{⎛} \{A_1^\Diamond, A_2^\Diamond, \ldots, A_n^\Diamond, O_{n+1}, \ldots, O_m, Q_{m+1}, ..., Q_p\}] \sum A^\heartsuit.$$

where $S(X,Y,Q)$ represents the didactic system created by the wish of a group of people X (the "inquirers") to approach question Q with the help of a group of supervisors Y and the milieu is made of a set of derived questions Q_i, labelled answers $A_n^\Diamond$ and other objects O_k.

We now use the billiards problem introduced in section 3.4 to illustrate the schema and to elaborate on the dialectic of the *media* and the *milieu* (Chevallard, 2008; Kidron, Artigue & Bosch, 2014).

4.2. The example of the billiards and the dialectic of questions and answers

We met the "Billiards Table Task" (Figure 2) during our own study of the literature on problem posing research. In order to better grasp the potential variations of this didactic situation, we undertook a study and research process which, in the end, appeared to us a rich illustration of the role played by *media* and *milieus* in the process of generating new questions and developing new answers. This section will first present a summary of the process we followed, describing its dynamics using the Herbartian schema just introduced.

We are not taking as generating question the exact statement of the Billiards Table Task as formulated in Figure 2. The situation proposed there belongs to an empirical study of research questions about students' skills and strategies in problem posing. As such, the situation is directly located in a mathematical context, formulated according to a specific research methodology and subject to various contextual constraints. In our case we are considering a more open generating question about what is known in billiards as a "bank shot" obtained when a ball bounces off an edge of a billiards table on its way to a pocket. Our generating question is (cf. Figure 3):

> Q_0: Given an initial position of the ball, how to get it into a pocket after bouncing off the table edge just once?

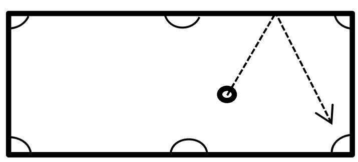


Figure 3. Billiards table and a "bank shot"

To investigate Q_0, the most evident experimental milieu is a real billiards table, some practice and possibly also some established answers obtained from experts in the billiards game, which can be found in publicly available media. If we do not have access to this experimental milieu, we can replace it with a more economical one: a graphical model drawn on a piece of paper (as in Figure 3). To make this milieu operational, that is to obtain some information from its manipulation, some previous knowledge needs to be assumed as part of the milieu, such as basic notions from elementary geometry (angles, isosceles triangles, etc.) and kinetics (linear movement of the ball, negligible friction, Newton's first and second law, and some elements of the physics of collisions). This knowledge is not strictly necessary in the first case since it can be replaced by interaction with the milieu: different trials and observation. In both cases, the initial question can be decomposed into three other ones (and certainly others not considered here):

Q_1: What is the initial trajectory of the ball after hitting it with the stick (cue)? What does it depend on? Can it be predicted? How?

Q_2: Which is the trajectory of the ball after bouncing off the edge of the table? What does it depend on? Can it be predicted? How?

Q_3: Given a pocket chosen as the target, where should the ball bounce off the edge of the table to hit it?

A direct consultation of experts or, in our case, a first look into the abundance of websites on billiard techniques, does give some answers to Q_1. If the cue hits the ball in the centre (as seen from the horizontal direction), the trajectory is a straight line; if it hits it off the centre, then there will be spin and the trajectory will not be linear. We note here that getting access to these pieces of answer (and also to some of the derived questions) requires one to get familiar with a number of objects, terms and techniques related to billiards: cue, cue ball (exceptionally absent in the case of bank shots), cushions, english (spin), bank and straight shots, etc. These are also small pieces of answer $A^\lozenge$ as just mentioned. Once these answers are processed and integrated into the milieu, new questions can be raised (see figure 4):

Q_{11}: How does the trajectory depend on the specific point where the cue hits the ball?

Q_{111}: Why does hitting the ball in the centre give a straight line, and hitting in off centre creates spin?

Q_{1111}: Is it the same phenomenon in football and other ball sports?

Q_{1112}: What is the scientific explanation of these phenomena?

Q_{112}: How precisely should the centre of the ball be hit to avoid spin and get a linear trajectory?

Q_{113}: How can spin be used to determine a non-linear trajectory?

Q_{114}: What different spin techniques do experts use?

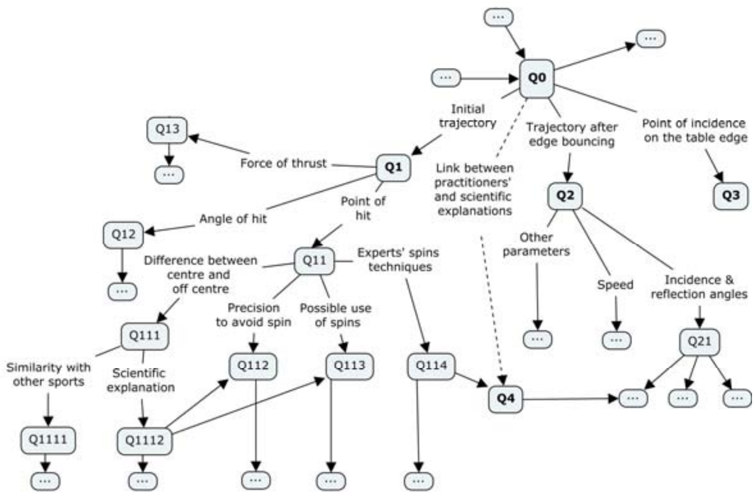


Figure 4: Tree of questions about the billiards bank shot (drawn up with CmapTools®)

To answer some of these questions, new media and milieus become necessary. One can go on asking experts, directly or through different media. Some of the questions (particularly Q_{111}) would probably require access to a different kind of source, where more technical explanations will appear, leading to new questions derived from Q_{111}, Q_{112}, Q_{113}, etc.

In particular, some answer can be found in academic works related to ball sports or to the physics of rotating bodies; or one may simply choose to ignore the causes and uses of spin because the generating question can be given a simpler answer when considering only straight paths of the ball. All the partial answers found will then need to be tested using the empirical and epistemic resources available in the milieu, giving rise to new questions and new search for answers. The process of this branch of the inquiry process can end when the provisional answer to Q_{11} is found to be satisfactory by the inquirers. But the answer will certainly leave some open issues, for instance a further questioning related to the relationships between the theoretical explanations and the practical techniques, a question that can be generalised to other cases and initiate a new main branch of the tree:

Q_{114}: How are the theoretical results about balls' motion and collisions used in the practice of billiards? Are they part of or related to the experts' "practical theory"? How?

Q_4: What is the billiards experts' "practical theory" made of? How is it related to the scientific explanations of the dynamics of collisions?

Question Q_2 will lead the inquiry to the study of other questions:

Q_{21}: What is the relationship between the angles of incidence and reflexion of the ball against the table edge?

Q_{211}: How is the relationship for a ball without spin?

Q_{212}: How is the relationship in the case of a spinning ball?

If we consider the case without spin, the answer to Q_{211} ("The intercept and reflexion angles are equal") can be considered a piece of established knowledge already available as part of the milieu and, as such, taken for granted. The study can thus go on without even noticing Q_{212}. In this case the potential question is ignored, as it is considered as an answer instead. However, this piece of established knowledge can be also questioned against this milieu: "We know (it seems that) they are equal, but we still want to know why". Suppose that the question is considered as such and that the considered milieu is not rich enough to produce an answer. Some external pre-established knowledge is necessary and, thus, new questions appear about how to get it. What kind of knowledge are we looking for? What labels can help identify it? Where to locate it? In what discipline or domain: game theory; geometry; physics; kinetics?

To deepen the study of Q_{211}, after obtaining the above answer, one can ask, for instance:

Q_{2111}: How can the identity between the incidence and reflection angles be used to get the ball in the pocket?

Q_{2111}: What does the identity between the incidence and reflection angles rely on? How can it be justified?

Q_{2112}: What are the limitations of the use and justification of the identity between the incidence and reflection angles?

Similar questions can be asked in the spinning case (if pursued) and the assumptions made there might put into question some of the assumptions of the non-spin case, with some productive comings and goings that we are not considering here.

4.3. Question-generation by the search and study of media

Let us consider in some detail a crucial issue in the nurturing of questions that has just been mentioned: finding appropriate information sources, or *media*, to access potentially useful answers.

Imagine that, in our approach to Q_{211} (identity between incidence and reflection angles), we find the following document posted on the web:

REFLECTING ON BILLIARDS

Academic Subjects: Physics, Geometry

Academic Topics: Reflection of light, reflection off cushions, congruent triangles

Background: The well-executed bank shot in billiards can appear as a matter of luck. Using the physics of mirrors, a player can substantially increase success.

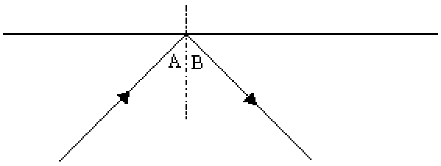
Discussion: A billiard ball hitting off a cushion will follow a symmetrical path. This is identical to the path that a beam of light will follow after hitting a mirror. The similarity allows one to better predict where a ball hitting a cushion will travel.

Figure 5: Introduction to a web document on billiards (ESPN Sports Figures, 2003)

We do not know the author nor the context this document belongs to (this could lead to new questions to pursue), but we cannot fail to perceive its didactical nature. It is clearly a *media*, with intentions to instruct: the text is labelled “Reflecting on billiards”, it indicates two academic disciplines (Physics and Geometry) and more specifically points to three themes within these disciplines: “Reflection of light”, “reflection off cushions”, “congruent triangles”. The two firsts labels seem to refer to Physics, the third one to Geometry. The short background given is also related to a problematic question – how to increase success in bank shots – which appears to be close to ours. The references to light and mirrors may indeed appear enigmatic in this context. The main information is given as a “discussion” which continues with the explanation in Figure 6.

The explanation then goes on to the case of a bank shot off of two cushions, similarly imagining both cushions as mirrors. It concludes with a final general statement which is an answer to Q_{21}: “Notice that the angle of incidence always equals the angle of reflection”.

The diagram below shows the path of a cue ball hitting a cushion or the equivalent path of a light beam hitting a plane mirror.

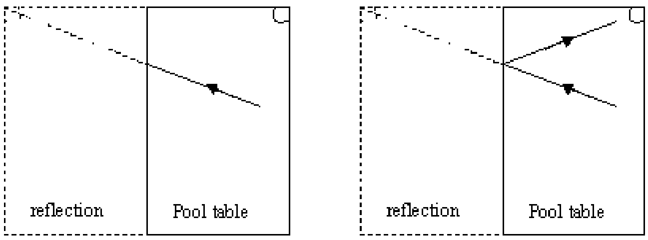


The dotted line in the diagram is called the normal line. It is a line drawn perpendicular to the reflecting surface.

The light ray and the normal to the surface define the angles of the light. They are named the angle of incidence and the angle of reflection. The angles are equal, as can be seen in the diagram.

[…]

Since a billiard ball follows the same path that the light does, we can imagine that the cushion of the pool table is a mirror. We can then aim at the reflection of the pocket in the reflection of the table.



The billiard ball will reflect off of the cushion and head for the pocket.

Figure 6: Continuation of the web document on billiards (ESPN Sports Figures, 2003)

As mentioned, we may simply trust the last answer (if we consider the media reliable enough) and proceed with the inquiry; or, a second and quick test against other media can be done, once we have located relevant key words. Accepting the answer, we thus integrate the identity between the angle of incidence and the angle of reflection into the milieu. The question leading the research process is then how to use the angles identity to solve question Q_{211}.

However, we can also start questioning the answers found in the text, and raise new questions to follow on with the process:

Q_{21121}: Is the analogy with light just something invented by the author to explain a billiard strategy, or is it a real technique used by expert billiard players when they shoot? Do experts really look at cushions as if they were mirrors?

Q_{21122}: Can the analogy with light be used as a justification of the identity between the incidence and the reflection angle?

We note here, in relation to the first question, that the document quoted in figures 5 and 6 is in fact the summary of a video of the American television series *Sports Figures* elaborated by ESPN (2003). The video provides answers to some of the abovementioned questions, and also raises new ones.

Regarding the second question, if the "physics of mirrors" is merely an analogy, we may look for firmer explanations that the angle of incidence is equal to the angle of reflection. This means going back to Q_{2112} and look for new pieces of answer, maybe searching more deeply in media like Physics textbooks or the web. In our case, we found only textbooks that do not treat the non-orthogonal collision of a ball and a wall. They do however something similar: the elastic collision of balls, where the preservation of impulse and kinetic energy can be used to calculate the velocity of the balls after collision, knowing the velocities before. This leads to the question of whether we can model the cushion as a ball which is initially at rest and which has infinite mass, and whether this can explain the phenomenon.

Pursuing instead Q_{21121} leads to research into a variety of billiard techniques including other basic strategies for bank shots, such as the use of the rows of "diamonds" that are on the edges of some billiard table. This gives rise to new questions about how these practical techniques might relate to basic geometrical results, especially when trying to explain why these techniques fail in some cases.

We invite the reader to pursue the inquiry, which can take on a clear mathematical dimension at this and other points. We now turn to some main characteristics of the dynamics of questions and answers, nourished by the dialectic of media and milieus, both illustrated by the above example.

4.4. The dialectic of media and milieu in the inquiry process

In Section 4.3, we have only outlined the beginning of some, among other possible, study and research paths which could be pursued from the initial generating question Q_0. In fact, there is no way we could have finished the diagram, even if writing a whole book. Indeed, the diagram ends with questions which could be generating questions for study and research paths in more specialised areas such as geometry, physics and the various techniques and theories of playing billiards.

This bootstrapping of the research and study process just means that it can go on as long as we want or until we are satisfied with the answers found. The decision of when to stop or pursue the study, which is closely related to the inquirers' consideration of what is an acceptable answer and what is not, appears as another important problem in the managing of research and study paths, a problem which is often minimised in the school context by means of the didactic contract: "a solution is acceptable when the teacher considers it to be so".

Of course, actual study and research processes are finite in time. In the Herbartian schema, the answer $A^{\heartsuit}$ constructed in response to a generating question Q_0 is a whole piece of work, a *praxeology* or an amalgam of praxeologies (Chevallard, 1999) including a practical dimension (new know-how) and a theoretical dimension (new knowledge, which could be more or less contingent on the study community). This work is also the result of a number of *choices* made by the study community: in the example, from the generating question Q_0 about billiards one could choose to deliberately pursue derived questions linked to school mathematics, without ignoring other questions or the work needed to give a reasonably practical answer to Q_0. Whatever choices are made, it must be clear that $A^{\heartsuit}$ should always appear as a partial answer to Q_0 so that the study and research path pursued does not degenerate into a random walk in school geometry…

As the billiards example illustrates, any study and research path contains an important amount of arbitrariness and is also subject to the contingency of the resources available: the inquirers' equipment in terms of knowledge and know-how, the possibility to count on a teacher (or a guide) to carry out the study, the different media within reach and their quality, the kind of milieus available, and the constraints on the project within the initial question is posed (time, kind of result expected). These conditions will contribute to shape the actual process, which concrete final result is thus difficult to predict.

It should also be noticed that, during the inquiry process, especially when it is not tightly guided, many detours and dead-ends will appear, with the examination of potentially useful answers that will be tested then finally discarded. A lot of informal knowledge and useless expertise may be constructed, then deconstructed, transformed or just set aside. There is no direct way from the question to a satisfactory answer, and there may not even be an optimal route to a certain $A^{\heartsuit}$. However, letting Q_0 be directive for the process and taking it seriously (not only as a mere excuse to learn some established answers and problem solving skills, or pursue any whimsy curiosity) can bring about criteria for the study community to make reasoned choices to focus on certain derived questions and answers, while discarding others.

In spite of the room for choice and erring, some invariants can be found. The most relevant here is the *dialectic of media and milieus* (Chevallard, 2008). It indicates how the interaction between new external answers accessed through media and the current objects and knowledge available in the milieu produce new knowledge about the questions approached and new issues to go on with the study. As has been seen in the example, the milieu is permanently nourished by the media via the new questions raised and the new pieces of information made available. This observation also suggests that the process of study and research might wither if it only relies on the limited resources of a given milieu: the evolution of the milieu, especially through the incorporation of new knowledge made available, is essential in the dynamics of the study and research process. At the same time, the access to media does not provide ready-to-use solutions, but just previously established answers which must be validated using the stable resources of the milieu before becoming usable for the ongoing inquiry. The search for available answers is, thus, another crucial dimension of the process. And neither of both seem to be encouraged enough in current educational contexts.

Finally, looking at the study and research process based on the Herbartian schema can help distinguish different strategies for raising new questions and thus for sustaining the dynamics of the inquiry:

(a) The first kind of questions considered are the initial *generating questions* considered as the starting point of the inquiry process. They are close to the idea of *driving questions* developed by Krajcik and Czerniak (1999). As suggested in Figure 4, they do not come out of the blue: they could come from earlier study and research paths developed by the same study community, or from other sources; but it certainly needs to have some links to the backgrounds, knowledge and aims of the study community.

(b) Some initial questions are derived from the generating question only by means of the resources available in the milieu: considering particular or general cases, identifying apparently pertinent variables, loosing or strengthening the conditions, etc. These kinds of questions are the main foci of research based on problem solving and posing.

(c) What is rarely pointed out is the way many of the derived questions come from the study of established answers, first to be found in media, and then tested against other media or against the milieu. This is how the inquiry process is nourished and enriched from the outside through the *dialectic of media and milieu.*

(d) Finally, some questions are raised during the elaboration of the final answer $A^{\heartsuit}$, the work on syntheses in the sense of Bouligand; such questions may focus on the scope of the questions produced (for instance: can they be generalised, or do we miss crucial conditions?), and connections between the pieces of answers produced during the inquiry and the already established knowledge. They are often a main source of questions that could not have been formulated before the completion of the path.

We now outline briefly how c. and d. above constitute missing links towards sustainable study and research processes in many established formats used in school mathematics.

4.5. The Herbartian schema and the four paradigms in section 2

Following Chevallard (2008), we can use the Herbartian schema and, especially, the media-milieu dialectic to analyse the most current pedagogical traditions in mathematics education.

Teaching *based on transmission* can be represented as the case where the initial question Q remains in the shadow ($\mathcal{Q}$) or is merely an artificial motivation proposed by the teacher to study an established answer $A^{\lozenge}$, such as congruence theorems for triangles. The deep and thorough appropriation of this answer – which could well include the study of a proof, and some main applications – results in a more or less personal version $A^{\heartsuit}$ which, however, should be as close to the teacher's proposal A^{Y} as possible, which, in turn, is supposed to agree with $A^{\lozenge}$; the process has a final goal and ends (ideally) with the beginning:

$$[S(X;\, Y;\, \mathcal{Q}) \left(\; \{ A^{Y}, O_{n+1}, \ldots, O_m \}] \sum A^{\heartsuit} = A^{Y} \approx A^{\lozenge}.$$

The teacher (possibly supported by one or more prescribed texts) is the main *media*, the official answer provided is not questioned nor tested against any milieu. It only needs to be incorporated in it for later use. The global teaching and learning process is defined by a set of established answers the students are supposed to integrate in their milieu with the help of the teacher.

In the paradigm of *learning by adaptation*, the only media is the teacher who ensures that there are no given questions or answers. The student has to produce his own questions and answers directly from a more or less natural milieu, a process which will usually end rather soon as the students' resources have been exhausted. Moving to new milieus produces new and independent questions and answers, unless the learner sees some direct similarities in the situations:

$$[S(X;\ Y;\ \mathbb{Q})\ \lceil\ \{O_1,\ \ldots,\ O_m,\ Q_{m+1},\ \ldots,\ Q_p\}]\ \Sigma\ A^{\heartsuit}.$$

In a *problem solving* situation, the initial question is crucial as a starting point of the process. However, apart from the already available milieu, there is usually no consideration of the possibility of searching in different media for already existing answers, nor of the process of *studying* these answers through a process of deconstruction and reconstruction.

$$[S(X;\ Y;\ Q)\ \lceil\ \{O_1,\ \ldots,\ O_m,\ Q_{m+1},\ \ldots,\ Q_p\}]\ \Sigma\ A^{\heartsuit}.$$

Problem posing situations can be considered a variant of problem solving, the initial question being to identify certain types of problems in a given context; the answer thus has to be given in the form of one or more problems. The dialectic of media and milieus is not taken into account: only the teacher can act (exceptionally) as media to clarify Q and give other forms of guidance, so the process has to proceed without any answers from the outside. Again, it stops as soon as the resources of the milieu have been exhausted.

The frame of didactic situations provides an example of a specific managing of the dialectic media-milieu, with a strict division of responsibilities between the teacher and the students which is at the base of the distinction between didactic and adidactic situations. It is clear that the starting point is an initial question Q, situated in an objective situation which is devolved to the group of students by a teacher. This objective situation contains an appropriate and evolving milieu made of established knowledge and stable objects, well known by the students to be used as experimental means in the building of an answer to Q. What about the already established answers and their access through the media? Everything is under the responsibility of the teacher in the pieces of information A_i^Y introduced in the milieu and also in the final institutionalisation of the answer obtained. This process has a definite beginning and end, both ensured and decided by the teacher.

$$[S(X;\ Y;\ Q)\ \lceil\ \{A_1^Y,\ \ldots,\ A_n^Y,\ O_{n+1},\ \ldots,\ O_m,\ Q_{m+1},\ \ldots,\ Q_p\}]\ \Sigma\ A^{\heartsuit} \approx A^{\Diamond}.$$

In all four cases, the theoretical impossibility of sustainable study and research processes (in the sense defined in Section 4.4) is thus, at least in part, caused by a double fact. The first one is that students are not expected or allowed to engage in a search for and study of established answers in media, as the media are limited to those provided by the teacher – and the teacher herself. The second one is that the answer $A^{\heartsuit}$

produced as the result of an inquiry process is not used for the development of new research and study processes.

5. THE HERBARTIAN SCHEMA AS A DESIGN TOOL: THREE OPEN QUESTIONS

The Herbartian schema has been proposed as a tool for the analysis of study and research processes. Through the description of mathematical themes or domains as a tree of questions and answers, it provides an alternative to the common vision of mathematical contents as a hierarchical structure of concepts and results. When it comes to the practical realization of sustainable research and study processes, some important questions remain open:

I. What are the didactic and mathematical infrastructures (and resources), as well as the associated knowledge, required for the design, monitoring and evaluation of sustainable study and research processes?

II. What are the institutional conditions needed for teachers to design and implement sustainable study and research processes, and for students to engage in them?

III. What kinds of constraints or even obstacles do institutions and societies commonly offer to such processes?

With regard to *didactic infrastructures* (that is, the material and knowledge equipment necessary to implement teaching and learning processes), there is an evident need for techniques related to the teachers' design and management of items like students' activities such as question posing, consultation of media, adaptation of public answers found to the questions examined, elaboration of syntheses of the solutions obtained, and students' means for self-monitoring their work with a set of generating questions as main or at least temporary leitmotiv. All these new teaching resources need to be based on a set of theoretical principles making a refined and coherent discourse for the teaching profession.

We also insist that the knowledge needed – and still largely to be developed – concerns not only the teachers' (and students') practice, but also the available *mathematical infrastructure*, that is, knowledge and resources of a purely mathematical nature. While working with designs and implementation of study and research processes, we frequently experience the shortcomings of our own mathematical viewpoints, generated mostly in the context of direct transmission and limited problem solving, mostly reduced to the "application" of given knowledge. In fact, the didactic transposition process is normally concerned with adapting *answers* $A^{\Diamond}$ to given institutions, learners and teachers, with questions appearing more in the background, as ways to test and promote the learners' digestion of these answers. We have much less techniques available to identify pertinent mathematical questions related to such answers, than to identify pertinent answers to a given question (assuming the question is manageable, which usually means sufficiently small and similar to previously encountered questions). Our mathematical epistemology is much richer when it comes to designate results, properties and objects as defined in answers than to describe and develop the questions which did or could lead to the answers, or be posed based on them. Mathematical theories are, to most users (with the sole and even very partial exception of mathematical researchers), a decorative background to smaller practices which, most of the time, appear quite isolated from each other. As a consequence, our relationship to mathematics is mostly *retrocognitive* (Chevallard, 2012), that is, oriented towards repeating, refining and applying previously encountered answers, rather than to raise, construct, study and answer new and exciting questions. While a retrocognitive relationship to mathematics can be sufficient and useful in many practical contexts, it is highly insufficient when it comes to engaging in sustained study and research processes, and even more for teachers who are to design and implement them. To do so, a different and richer mathematical practice is needed, in particular, *networks of crucial questions* should become explicit parts of the theoretical structure, as the few existing epistemological analyses based on the Herbartian schema have to some extent evidenced. But by and large, these new elements of mathematics still have to be developed by researchers and for, with and by teachers.

The second question is mainly related to the peculiar expression "for, with and by teachers". Perhaps institutional conditions evoke, to many readers, above all external requirements and resources found in school institutions, which could be more or less conducive for the admittedly ambitious goal of engaging students in sustainable study and research processes; these external factors should certainly not be ignored. But the role of teachers is also very important, at least in our limited experience which varies from content driven, strongly time constrained contexts like classical university courses, to primary school teaching with considerable liberty for teachers, strong incentives for student inquiry and so on. Our strategy for approaching this second question would therefore be to consider the *interaction* between institutional conditions and the development of the teacher profession (Cirade, 2006) with the consequent shared knowledge of teachers. We have seen that an isolated teacher, supported by external partners such as researchers or principals, may be able to develop and adapt a successful research and study process design to even rather difficult conditions; but the design may not lead to a lasting (or indeed, sustainable) didactic practice without all these extraordinary conditions. In particular, another teacher who tries to adopt the design may more or less involuntarily revert to other, usually less demanding, forms of teaching, as found by Barquero et al. (2013). Teachers' professional knowledge can only become stable to the extent that institutional conditions are available for shared development, dissemination and validation of such knowledge, as has been argued by several authors (e.g. Stigler & Hiebert, 1999; Chevallard, 2005). As proposed by Winsløw (2011), citing the Japanese framework of lesson study as a remarkable example, we can talk of necessary *paradidactic infrastructures*, that is: conditions existing in school and society for teachers to develop a professional knowledge based in close cooperation with researchers. Thus, the second of the above questions concerns the infrastructural needs outside of the classroom (in schools, and with external partners); the question seems crucial and entirely open. For instance, lesson study in the usual sense might be insufficient here, since it is focused on the planning, observation and analysis of single lessons; but research pursuing the second question may still consider the use of lesson studies to develop shared and refined didactic knowledge on more local aspects of study and research processes, or to develop a more refined knowledge of how a mathematical question can be developed (to enrich a first desktop analysis like the one given in Sections 4.2 and 4.3).

The third question of obstacles from school and society is clearly related to the study of existing didactic transpositions, since expectations and knowledge related to teaching will largely be founded in them – both for teachers, parents, politicians and other social partners of the school institution. In ATD, the study of higher levels of constraints on school teaching (Chevallard, 2002) is only emerging even for the analysis of the most stable forms of mathematics teaching; but at least it provides some first handles and indications for this problem. For instance, new paradidactic infrastructures could meet obstacles at the level of the discipline itself: is it really feasible and reasonable that mathematics teachers would have to develop new mathematical organisations of contents? Which is the role of researchers (including mathematicians) in this enterprise? Besides, other maybe stronger (because invisible) obstacles will appear at the levels of school and society: school curricula based on contents to be known and on rigid monodisciplinary organisations of these contents; role of the school as privileged means to access reliable knowledge (school "monopole" of the access to "good knowledge"); difficulties in developing a Herbartian attitude and, in terms of Schein (2013, p. 3) "become better at asking and do less telling in a culture that overvalues telling". It seems to us that approaching this kind of cultural and societal obstacles could not be done without taking into account a variation of institutional, societal and cultural contexts (Artigue & Winsløw, 2010). Investigating these obstacles more thoroughly remains largely work to be done.

We consider that these three questions are, in their full sense, entirely open, even if the first and the last questions have been investigated under very specific, local conditions, with interventions based on a thorough *a priori analysis* which is strongly affected by those conditions. For instance, Barquero et al. (2013) present an account of a full first course of mathematics for first year natural science university degrees based on the generating question of populations growth; García et al. (2006) consider the question of proportionality as the answer, between many others, to the problem of describing different possible saving plans, as worked with by a secondary school class; Jessen (2014) studied third year high school students' projects combining mathematics and biology around drug dosing; Parra, Otero and Fanaro (2013) connected the model of offer and supply in microeconomics with the topics "linear functions", "straight lines", "systems of equations" and "limits and derivatives"; finally, Sierra (2006) provides a completion of Brousseau's didactic engineering work on enumeration and measurement in terms of sequences of questions and praxeologies of increasing complexity. In all these cases, different *study and research paths* based on the Herbartian schema have been experimented in different institutional settings: primary, secondary, tertiary and teacher education. However, in all cases, the experimental conditions were very limited, since the implementation of study and research paths was closely monitored by researchers, who also frequently took part in the teaching.

In summary, previous research allows only very local conclusions about the first and the last question above, while the second question remains unanswered except for the very special cases when researchers participate directly in the planning and, frequently, also in the teaching. Even in these cases, it also appears that the ecology of study and research processes is fragile, in part due to the troubles for the students to accept the new didactic contracts that are needed, and also to the difficulties for the teachers not to give in to the students' demands and move back to the usual didactic contract: asking questions instead of requesting them, limiting the access to media, immediately assessing the answers obtained, giving detailed indications about the paths to follow, etc. In a way, this only adds to the interest of the full questions above, as some of these difficulties could be hoped to disappear in the case of experiments at institutional scale, with *all* teachers and students within a school institution being engaged over a longer period of time. And, as a complementary strategy to research based on more or less ambitious interventions, comparative studies across cultures, institutions and didactic stakes are also needed to gain full insight into the three questions.

When it comes to didactic stakes, these are often heavily compartmentalized in disciplines. While both scientific and school disciplines are to some extent characterized by specific objects and forms of inquiry, comparison, inspiration and combinations across disciplinary boundaries are necessary to develop new and better forms of inquiry at school. For instance, we can find interesting investigations in science education, where the activity of problem posing and solving is often more directly motivated by a need to explore and explain crucial questions about the world. For instance, the research on “problematisation and conceptualisation” promoted by Orange (2005) and others is based on a deep analysis of the dynamical relationship between problems and bodies of knowledge, as well as on an explicit conception of “building necessities” through study and research, making scientific knowledge appear as more or less compelling answers to the problems.

Our choice to explore the simple epistemological model proposed by Bouligand in terms of *syntheses* and *problems,* aimed at highlighting how the production, deconstruction and reconstruction of syntheses – or available answers in the terms of the Herbartian schema – constitute a driving force of the inquiry process. As such, they need to be incorporated in the epistemological frame adopted by didactics research. In the first part of the paper we saw the shortcomings, in this respect, of historic approaches to teaching and learning phenomena. In the second part of this paper, we have shown through a case study how the Herbartian schema from ATD can accommodate the dialectics proposed by Bouligand. In the last few years, it has in fact been productively used to design, experiment and analyse different types of teaching and learning processes. Of course this young frame is still to be developed and in this paper we have put it into a wider perspective to help relate it to theoretical and empirical studies done in other fields.

REFERENCES

ARTIGUE, M., BLOMHØJ, M. (2013). Conceptualizing inquiry-based education in mathematics, *ZDM 45,* 797-810.

ARTIGUE, M., WINSLØW, C. (2010). International comparative studies on Mathematics Education: A viewpoint from the Anthropological Theory of Didactics. *Recherches en Didactique des Mathématiques 31*(1), 47-82.

BARQUERO, B., BOSCH, M., GASCÓN, J. (2013). The ecological dimension in the teaching of mathematical modelling at university. *Recherches en Didactique des Mathématiques* 33(3), 307-338.

BARQUERO, B., BOSCH, M. (2015). Didactic engineering as a research methodology: from fundamental situations to study and research paths. In A. Watson, M. Ohtani (Ed.), *Task Design In Mathematics Education. An ICMI study 22*, Springer.

BOSCH, M., GASCÓN, J., RUIZ-OLARRÍA, A., ARTAUD, M., BRONNER, A., CHEVALLARD, Y., CIRADE, G., LADAGE, C., LARGUIER, M. (2011, Eds), *Un panorama de la TAD. An overview on ATD*. Barcelona: Centre de Recerca Matemàtica.

BOULIGAND, G. (1957). L'activité mathématique et son dualisme. *Dialectica 11*, 121-139.

BROUSSEAU, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publishers.

BROWN, S., WALTER, M. (1983). *The art of problem posing*. Philadelphia: Franklin Institute Press.

CAI, J., MOYER, J. C., WANG, N., HWANG, S., NIE, B., GARBER, T. (2013). Mathematical problem posing as a measure of curricular effect on students' learning. *Educational Studies in Mathematics*, 83:57–69.

CHEVALLARD, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques,* 19(2), 221-266.

CHEVALLARD, Y. (2002). Organiser l'étude : 3. Ecologie & régulation. *XIe école d'été de didactique des mathématiques* (pp. 41-56). Grenoble: La Pensée Sauvage.

CHEVALLARD, Y. (2005). Didactique et formation des enseignants. In B. David (Ed.), *Impulsions 4*, 215-231. Lyon: INRP.

CHEVALLARD, Y. (2008). Un concept en émergence: la dialectique des média et des milieux. In G. Gueudet, Y. Matheron (Eds), *Actes du séminaire national de didactique des mathématiques, année 2007* (pp. 344-366). Paris: ARDM & IREM de Paris 7.

CHEVALLARD, Y. (2015). Teaching mathematics in tomorrow's society: a case for an oncoming counterparadigm. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education*, Springer.

CHIN, C. (2002). Open investigations in science: posing problems and asking investigative questions. *Teaching and Learning 23*(2), 155-166.

CIRADE, G. (2006). *Devenir professeur de mathématiques: entre problèmes de la profession et formation en IUFM* (Doctoral dissertation). Université de Provence, France.

DIENES, Z. (1963). On the learning of mathematics. *The Arithmetic Teacher* 10 (3), 115-126.

ESPN (2003). *Sport Figures*. "Reflecting on billiards" http://www.hawleys.com/videos/reflecting_on_billiards.wmv and

FABRE, M. (1997). Pensée pédagogique et modèles philosophiques: cas de la situation-problème. *Revue Française de Pédagogie* 120, 49-58.

GARCIA, F.-J., GASCON, J., RUIZ-HIGUERAS, L., BOSCH, M. (2006). Mathematical modelling as a tool for the connection of school mathematics, *ZDM* 38(3), 226-246.

HALMOS, P. (1980). The heart of mathematics. *The American Mathematical Monthly* 87 (7), 519-524.

JESSEN, B. (2014). How can research and study courses contribute to the teaching of mathematics in an interdisciplinary setting? *Annales de Didactique et de Sciences Cognitives* 19, 199-224.

KIDRON, I., ARTIGUE, M., BOSCH, M. (2014). Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD. In A. Bikner-Ahsbahs, S. Prediger (Eds.) *Networking of Theories as a Research Practice in Mathematics Education* (pp. 153-177). Dordrecht: Springer.

KILPATRICK, J. (1987). Problem formulating: where do good problems come from? In A. Schoenfeld (Ed.): *Cognitive science and mathematics education* (pp. 123–147). Hillsdale, NJ: Erlbaum.

KONTOROVICH, I., KOICHU, B., LEIKIN, R., BERMAN, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *Journal of Mathematical Behavior, 31*, 149-161.

KRAJCIK, J., CZERNIAK, C. (1999). *Teaching Science in Elementary and Middle School: A Project-Based Approach*. New York: McGraw-Hill.

LEUNG, S. (2013). Teachers implementing mathematical problem posing in the classroom: challenges and strategies. Educational Studies in Mathematics 83 (3), 103-116.

MAROU, H. (1956). *A history of education in Antiquity*. Madison: U. of Wisconsin Press.

MIYAKE, N., NORMAN, D. (1979). To ask a question, one must know enough to know what is not known. *Journal of Verbal Learning and Verbal Behavior* 18 (3), 357–364.

NCTM (2000) *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

ORANGE, C. (2005). Problématisation et conceptualisation en sciences et dans les apprentissages scientifiques. *Les sciences de l'éducation - pour l'ère nouvelle 38*, 69-94.

PARRA, V., OTERO, M.-R., FANARO, M.-A. (2013). Recorridos de Estudio e Investigación co-disciplinares a la Microeconomía. *Números 82*, 17-35.

PÓLYA, G. (1945). *How to solve it: A new aspect of mathematics method*. Princeton, NJ: Princeton University Press.

ROUSSEAU, J. J. (1762). *L'Émile, ou de l'éducation.* The Hague : Librairie Jean Néaulme.

SCHEIN, E. (2013). *Humble inquiry. The gentle art of asking instead of telling*. San Francisco, CA: Berret-Koehler.

SCHOENFELD, A. H. (1985). *Mathematical problem-solving*. Orlando, FL: Academic Press.

SCHOENFELD, A., KILPATRICK, J. (2013). A US perspective on the implementation of inquiry-based learning in mathematics, *ZDM 45,* 901-909.

SIERRA, T.-A. (2006). *Lo matemático en el diseño y análisis de organizaciones didácticas. Los sistemas de numeración y la medida de magnitudes* (Doctoral dissertation). Universidad Complutense de Madrid. http://eprints.ucm.es/tesis/edu/ucm-t29075.pdf

SINGER, F., ELLERTON, N., CAI, J. (2013). Mathematical Problem Posing. *Educational Studies in Mathematics 83*, 1–7

SINGER, F., VOICA, C. (2013). A problem solving conceptual framework and its implications in designing problem posing tasks. *Educational Studies in Mathematics* 83(3), 9-26.

STIGLER, J., HIEBERT, J. (1999). *The Teaching Gap*. New York: The Free Press.

WINSLØW, C. (2011). A comparative perspective on teacher collaboration: the cases of lesson study in Japan and of multidisciplinary teaching in Denmark. In: G. Gueudet, B. Pepin, L. Trouche (Eds), *Mathematics curriculum material and teacher documentation: from textbooks to shared living resources* (pp. 291-304). New York: Springer.

WINSLØW, C., MATHERON, Y., MERCIER, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics* 83 (2), 267-284.

WINSLØW, C. (to appear). ATD and other approaches to a classical problem posed by F. Klein. To appear in the *Proceedings of the fourth international congress on research in ATD* (Toulouse, France).